Spatially Homogeneous Solutions to the Einstein-Maxwell Equations

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Two spatially homogeneous line elements, representing two inequivalent solutions to the Einstein-Maxwell equations, are known at present. The relationship between the two line elements is pointed out and discussed briefly here.

THREE years ago, the author derived a spatially homogeneous solution to the Einstein-Maxwell equations. The line element associated with the solution, Eq. (4.10) in the published work,[1] is

$$(ds)^2=-\frac{b_1{}^2(\tan\frac{1}{2}t)^{2(b_2+b_3)}}{(\sin t)^4}(dt)^2+(\sin t)^2(dx_1)^2 +\frac{(\tan\frac{1}{2}t)^{2b_2}}{(\sin t)^2}(dx_2)^2+\frac{(\tan\frac{1}{2}t)^{2b_3}}{(\sin t)^2}(dx_3)^2,\quad [b_2b_3=1]. \tag{1}$$

Two independent constants appear in (1), b_1 and either b_2 or b_3 $(=1/b_2)$, and they provide an intrinsic parametrization of the solution.[2] For the solution represented by the line element (1), $t=$constant hypersurfaces are metrically Euclidean with disposable topology, and the fundamental Rainich geometry scalar invariant α is identically constant, i.e., independent of the time coordinate t. If we specialize to the case $b_2=b_3=\pm1$, then the line element (1) reduces to a form symmetrical with respect to rotations around the x_1 axis, namely,

$$(ds)^2=-\frac{b_1{}^2(dt)^2}{(1\pm\cos t)^4}+(\sin t)^2(dx_1)^2 +\frac{(dx_2)^2+(dx_3)^2}{(1\pm\cos t)^2},\quad [b_2=b_3=\pm1]. \tag{2}$$

Without reference to the form (1), Brill[3] has recently published a spatially homogeneous solution to the Einstein-Maxwell equations. By combining Brill's Eqs. B(1), B(2), and B(20),[3] his line element is given by

$$(ds)^2=-(4B_0{}^2/A^2)(dt'')^2+A^2\sigma_z{}^2+B^2(\sigma_x{}^2+\sigma_y{}^2), \tag{3}$$

where σ_x, σ_y, σ_z are Cartan differentials for the spatial coordinates in a 3-dimensional spherical space, satisfying conditions B(3)

$$d\sigma_z=\sigma_x\wedge\sigma_y \text{ and cyclically}, \tag{4}$$

A and B in (3) are certain algebraic functions of t'', and B_0 is a constant. For the solution represented by the line element (2), $t''=$constant hypersurfaces are 3-spheres metrically and therefore topologically, and the fundamental Rainich geometry scalar invariant α depends on the time coordinate t''.

We may establish a relationship between the line elements (2) and (3) by letting the radius of the spatial 3-spheres associated with (3) increase without bound, so that the spatial part of (3) becomes Euclidean, like the spatial part of (2). First we renormalize the Cartan differentials σ_x, σ_y, σ_z with a change of scale, introducing a constant reference radius for the spatial 3-spheres, say R_0, by means of the scale transformation: $\sigma_x\to\sigma_x/R_0$, $\sigma_y\to\sigma_y/R_0$, $\sigma_z\to\sigma_z/R_0$. Under the scale transformation, we have $A\to R_0A$, $B\to R_0B$, $B_0\to R_0B_0$ in order for the line element (3) to remain invariant. Also, the conditions (4) become

$$d\sigma_z=(1/R_0)\sigma_x\wedge\sigma_y \tag{5}$$

and cyclically. Then, by letting R_0 increase without bound, (5) gives Cartan's conditions for a Euclidean spatial metric

$$d\sigma_x=d\sigma_y=d\sigma_z=0 \tag{6}$$

and the line elements (2) and (3) are found to be equivalent by putting

$$\begin{aligned} b_1/(1\pm\cos t)&\equiv 2B_0t''+C_0,\\ dx_1&\equiv\sigma_z,\\ dx_2&\equiv D_0\sigma_x,\\ dx_3&\equiv D_0\sigma_y, \end{aligned} \tag{7}$$

in which C_0 and D_0 are extra constants. Thus, from (2) we obtain the algebraic functions of t'' in (3),

$$A^2=\frac{2b_1}{2B_0t''+C_0}-\left(\frac{b_1}{2B_0t''+C_0}\right)^2, \tag{8}$$

$$B^2=(D_0/b_1)^2(2B_0t''+C_0)^2. \tag{9}$$

The latter expressions have the same algebraic form as Brill's Eqs. B(21) and B(22), differing only with regard

[1] G. Rosen, J. Math. Phys. 3, 313 (1962).

[2] Interest has recently been attached to solutions which may be continued analytically through spurious singularities (due to a singularity in the coordinate system) to give qualitatively different space-time geometries [for example, see C. W. Misner, J. Math. Phys. 4, 924 (1963)]. The temporal singularities in the line element (1), singularities appearing at $t=n\pi$ with n an integer, are however intrinsic to the geometry (Ref. 1). From an initial instant of coordinate time $t=t_0$ such that $(n-1)\pi<t_0<n\pi$, the geometrically singular state at $t=n\pi$ is reached in a finite duration of *proper time* if n is even (odd) and b_2 is positive (negative), representing an unstable world with a finite proper lifetime, while on the other hand, the geometrically singular state is *not* reached in a finite duration of *proper time* if n is odd (even) and b_2 is positive (negative), representing a stable world with an infinite proper lifetime. No analytic continuation through a singularity to a new, qualitatively different, space-time geometry of physical interest is possible here.

[3] D. R. Brill, Phys. Rev. 133, B845 (1964). Equations numbered with a B are found in this paper.

to constants of integration.[4] By making obvious scale transformations, B_0, C_0, and D_0 may be eliminated from the line element, b_1 surviving to parametrize the solution.

Hence, the line element (2), representing a solution which is obtained either by specializing the author's line element (1) or by specializing Brill's line element (3), is common to both spatially homogeneous solutions to the Einstein-Maxwell equations, even though the two solutions are generally inequivalent. On the one hand, the author's line element (1) represents a more general solution that is not necessarily symmetrical with respect to rotations about a preferred axis (the x_1 axis). On the other hand, Brill's line element (3) represents a more general solution that is not necessarily Euclidean in the hypersurfaces of constant time, but symmetrical with respect to rotations about a preferred axis (the z axis). The fact that the two spatially homogeneous solutions were obtained originally from very different formulations of the Einstein-Maxwell equations, using very different integration procedures, illustrates the value of our having and studying alternative formulations of field equations in general relativity, such as the Rainich and Cartan formulations of the Einstein-Maxwell equations.

[4] The difference in the constants of integration is mainly due to the limiting procedure $R_0 \to \infty$, but also due in part to the fact that B(21) is not the *general solution* of the nonlinear total differential equation

$$d^2B/dt''^2 = B_0^2/B^3$$

supplemented with the initial condition $B=B_0$ at $t''=t_0''$, the general solution being $B^2=B_0^2+2kB_0(t''-t_0'')+(1+k^2)(t''-t_0'')^2$ with k a new and free constant of integration; Eq. B(21), the *particular solution* with $k=0$, is not as general as it ought to be.

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Origin of the Light Elements

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It is shown that the observed isotopic abundances of Li, Be, and B can be explained by their spallation in small, prototerrestrial bodies. Spheres of arbitrary composition and radius are irradiated by protons; approximate expressions are found for the solar-flare proton spectrum, the spallation cross sections, and neutron production. A new approximation is made for the effect of the fast neutrons. It is then found that the present day proton flux is too soft to give the desired results reasonably, and that a mean proton energy of 300 MeV is necessary to get the observed isotopic ratios. The results are not sensitive to the composition, and we can obtain the measured Li, Be, and B abundances by taking dry silicate spheres of about 140 m for the proto-asteroidal bodies. In order to obtain the observed D/H ratio from the irradiation, however, it is necessary to add 10% H_2O. The measured crustal abundances of Li, Be, B lead to different values for D/H and for the depletion of Gd^{157} for the earth and for asteroids, contrary to observation. These discrepancies disappear if we assume that Li, Be, and B have been concentrated tenfold in the earth's crust. The different isotopic ratios found for terrestrial and meteoritic material are consistent with this model, and enable us to calculate the Li^7/Li^6 ratios to be expected on the other planets.

INTRODUCTION

THE origin of the light elements H^2, Li, Be, and B has long been a puzzle to cosmologists.[1,2] It has been suggested that they were produced by spallation reactions[2] on heavier elements. A reasonable such model was proposed by Fowler, Greenstein, and Hoyle[3]: Energetic solar protons spallated the light isotopes in solid bodies, and neutrons, simultaneously produced, bathed the resulting nuclei, tending to create the observed abundances.

The astrophysical setting posited by FGH is also assumed here: A rather cool protosun extremely active in emitting energetic protons, the solar system outgassed and inhabited by relatively small solid objects (protoplanets or planetesimals) orbiting the protosun and being irradiated by it.

There are two principal differences between the calculation presented in this paper and the one undertaken by FGH: First, they work backward from the presently observed isotopic abundances to an inferred "intermediate stage" in the evolution of the solar system. Here, we start from several plausible intermediate stages and follow the results of the solar proton bombardment forward. Secondly, FGH uses the mean value

[1] R. A. Alpher and R. C. Herman, Rev. Mod. Phys. 22, 153 (1950).
[2] E. M. Burbidge, G. R. Burbidge, W. A. Fowler, and F. Hoyle, Rev. Mod. Phys. 29, 547 (1957); W. A. Fowler, G. R. Burbidge, and E. M. Burbidge, Astrophys. J. Suppl. 2, 167 (1955). S. Bashkin and D. C. Peaslee, Astrophys. J. 134, 981 (1961).
[3] W. A. Fowler, J. L. Greenstein, and F. Hoyle, Geophys. J. 6, 148 (1962)—hereafter referred to as FGH; Am. J. Phys. 29, 393 (1961).